

Cycles and Clustering in Multiplex Networks

Gareth J. Baxter,^{1,*} Davide Cellai,^{2,3} Sergey N. Dorogovtsev,^{1,4} and José F. F. Mendes¹

¹*Department of Physics & I3N, University of Aveiro, Portugal*

²*Idiro Analytics, Clarendon House, 39 Clarendon Street, Dublin 2, Ireland*

³*MACSI, Department of Mathematics and Statistics, University of Limerick, Ireland*

⁴*A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*

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In multiplex networks, cycles cannot be characterized only by their length, as edges may occur in different layers in different combinations. We define a classification of cycles by the number of edges in each layer and the number of switches between layers. We calculate the expected number of cycles of each type in the configuration model of a large sparse multiplex network. Our method accounts for the full degree distribution including correlations between degrees in different layers. In particular, we obtain the numbers of cycles of length 3 of all possible types. Using these, we give a complete set of clustering coefficients and their expected values. We show that correlations between the degrees of a vertex in different layers strongly affect the number of cycles of a given type, and the number of switches between layers. Both increase with assortative correlations and are strongly decreased by disassortative correlations. The effect of correlations on clustering coefficients is equally pronounced.

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The realization that many complex systems cannot be understood by representing them as a single network, has led to an explosion of interest in multilayer and multiplex (multiple types of edges) networks. Applications range from infrastructure [1], financial [2], transport [3] and ecological [4].

To properly study multilayer systems, it is essential to understand the fundamental properties of such structures. Many concepts from single layer networks have already been generalized to multiple layers, from the degree distribution, to connectivity, adjacency and Laplacian matrices, centrality measures and so on [5–7]. In many cases, the generalization of concepts from single networks—for example, the meaning of “giant connected component” [8, 9]—is not straightforward, and introduces a new dimension to the problem.

In this Letter, we give an analytical description of the statistics of cycles in multiplex networks. In particular, we consider multiplex networks characterized by a given multi-degree distribution (configuration model). As this model is the starting point of any network analysis of a real-world system, it is easy to see how important is to characterize analytically a structural property like the statistics of cycles. In a multilayer network, the possibility to switch between layers greatly increases the number of cycle types with respect to the single layer case. Cycles are then no longer defined simply by their length. We must take into account the proportion of the cycle in each layer, as well as the number of switches between layers. In particular, this leads to a set of different clustering coefficients generated by different cycles of length 3.

Even when two cycles contain the same number of edges of each color, they can differ in the way colored edges are arranged within the cycle. We introduce a vec-

tor l characterizing the number of edges of each color, and the number of switches between layers of a cycle. We give formulae for calculating the mean number of cycles corresponding to a given l in a random graph with a given size and degree distribution. As examples, we calculate the distribution of edge colors and switches in cycles of a given length, show the effects of degree correlations between layers, and examine the special case of cycles of length 3, which allows the calculation of the generalized clustering coefficients in multiplex networks.

The statistics of cycles is relevant both from a theoretical and an applicative point of view. From a theoretical perspective, it allows one to understand whether the distribution of cycles observed in a real world network is significantly different from that in a random graph with similar statistics [10]. Even in the single layer case, the high concentration of finite cycles in real-world networks has been a formidable barrier to analytic treatment, as mathematical models of large networks are typically based on the local tree-like assumption, i.e. the vanishing of density of finite cycles as the size of the network diverges [11–13]. On the other hand, some analytic theories have been quite successful even in real-world networks [14], suggesting that the role played by the detailed architecture of topological correlations and cycles is not easily characterized. In multiplex networks, further types of correlations arise naturally, with implications for the structural properties of the multiplex [15, 16]. It is therefore necessary to turn now to multiplex cycles and investigate the statistics of each category based on edge color and layer switches, and the effects of correlations on them. The properties of cycles is also an important topic of graph theory [17, 18]. A relatively smaller volume of work has been devoted to cycles on colored edge graphs, mainly involving only theorems of existence [19].

The statistics of cycles in multiplex networks is also relevant for a number of applications. In information technology, the presence of multiple paths of heterogeneous colors is a structural property that improves the robustness of a network [20] and the security of a wireless sensor network from a malicious attack [21]. Multiplex cycles are also relevant when examining commuter behavior on multiple transport networks, as they provide alternative routes [3]. In applications such as this, switching between layers may have a time or monetary cost, a consideration which is absent in single layer networks. The statistics of switches is therefore an essential part of any analysis of cycles in multiplex networks.

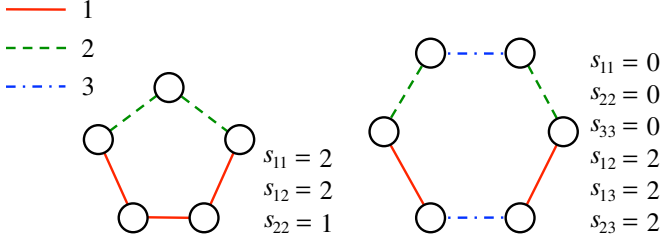


FIG. 1. Notation of multiplex cycles.

We characterize a given cycle on a multiplex network, by the vector $l = \{s_{ab}\}_{a,b=1,\dots,M}$, where each element s_{ab} defines the number of nodes in the cycle which connect an edge of type a with an edge of type b . When $a = b$, this counts the number of nodes where the cycle remains in the same layer. When $a \neq b$, s_{ab} counts the number of switches between layer a and layer b . In this Letter, we identify and study the classes of equivalence of cycles with the same l . In addition, here we only consider cycles without an orientation, therefore the order of the switches is not important, i.e. $s_{ba} = s_{ab}$. Examples of cycles with a given l are shown in Fig. 1. As a consequence of this definition, the total number of edges in layer a is then $n_a = s_{aa} + \frac{1}{2} \sum_{b \neq a} s_{ab}$. Clearly the numbers of switches must be such that n_a is an integer for all a . The total length of a cycle L is the sum of all entries of l .

We consider a generalization of the configuration model to multiplex networks, i.e. large sparse random multiplex networks with N nodes in M layers, defined by the joint multidegree distribution $P(q_1, q_2, \dots, q_M)$. This ensemble includes all possible configurations with multidegree sequence sampled from this distribution with equal statistical weight [22, 23].

To calculate the mean number of cycles $\mathcal{N}(l)$ with a given l in a random graph, we first count the number of ways we can select, from the given multidegree distribution, the nodes that have the connectivity required for each s_{ab} ; then, we count the number of ways we can connect these nodes to form a cycle. This method is similar to that used in, for example, Ref. [24] but we extend it to account for edges of different colors and switches between layers. The results can be written as the product

of several factors:

$$\mathcal{N}(l) = G(l)W(N, l)R[N, l, P(\mathbf{q})], \quad (1)$$

where $R(N, l, P(\mathbf{q}))$ counts the number of ways one can select L nodes, having the correct sequence of multidegrees that matches the elements of vector l . $W(N, l)$ counts the number of graphs in the ensemble containing the cycle, and $G(l)$ counts the number of ways of arranging the selected nodes to form a cycle. In principle one can complete this calculation for an arbitrary cycle in a multiplex with an arbitrary number of layers. However, the calculation of $G(l)$ becomes somewhat complicated for more than two layers, for anything but the shortest cycles.

Statistics of cycles.—Let us focus, now, on a two layer random multiplex (duplex), defined by the joint degree distribution $P(q_1, q_2)$. In the case of two layers, l has three entries: s_{11} , s_{22} , and s_{12} . For a given L , s_{11} and s_{22} can take any value from 0 to L , while the number of switches $s_{12} \equiv 2p$, for integer p , to ensure that the number of edges of each color is integral, while all three must satisfy $s_{11} + s_{22} + s_{12} = L$.

The formula (1) can be calculated explicitly in two asymptotic cases, both characterized by $L \ll N$, where $L \rightarrow 1$ and $1 \ll L \ll N$, respectively. In the first case, the probability of selecting a given set of nodes is asymptotically the same as if the selection was made with replacement, that is, each node's degree probability is unaffected by the other nodes. This leads to the simplified formula:

$$\mathcal{N}(l) = \binom{s_{11} + p - 1}{s_{11}} \binom{s_{22} + p - 1}{s_{22}} \frac{1}{2p} \left[\frac{\langle q_1(q_1 - 1) \rangle}{\langle q_1 \rangle} \right]^{s_{11}} \times \left[\frac{\langle q_2(q_2 - 1) \rangle}{\langle q_2 \rangle} \right]^{s_{22}} \left[\frac{\langle q_1 q_2 \rangle}{\sqrt{\langle q_1 \rangle \langle q_2 \rangle}} \right]^{2p}, \quad (2)$$

where $\langle \dots \rangle$ indicates averages with respect to the degree distribution (ensemble averages), for $s_{12} > 0$ or, when the cycle is only in a single layer

$$\mathcal{N}(L, 0, 0) = \frac{1}{2L} \left[\frac{\langle q_1(q_1 - 1) \rangle}{\langle q_1 \rangle} \right]^L, \quad (3)$$

and similarly for $\mathcal{N}(0, L, 0)$, exchanging subscripts. This coincides with the single layer result found in Ref. [24].

For larger cycles, but still short compared with the number of nodes in the network, we instead sum over all possible sets of nodes of each degree, using a delta function to ensure the total number of nodes is correct. The integral representation of these delta functions can be approximated using the saddle point method. We find

that, when none of s_{11}, s_{22}, s_{12} is zero,

$$\begin{aligned} \mathcal{N}(l) = & (2p-1)! \frac{(s_{11}+p-1)! (s_{22}+p-1)!}{(p-1)! (p-1)!} \\ & \times \frac{e^L}{(2\pi)^{3/2} \sqrt{s_{11}s_{22}s_{12}}} \langle q_1 \rangle^{-s_{11}-p} \langle q_2 \rangle^{-s_{22}-p} \\ & \times \left[\frac{\langle q_1(q_1-1) \rangle}{s_{11}} \right]^{s_{11}} \left[\frac{\langle q_2(q_2-1) \rangle}{s_{22}} \right]^{s_{22}} \left[\frac{\langle q_1 q_2 \rangle}{s_{12}} \right]^{s_{12}}. \end{aligned} \quad (4)$$

In the case $s_{12} = 0$, the cycle consists of only one color, so $\mathcal{N}(l) = 0$ unless either $s_{22} = 0$ or $s_{11} = 0$, in which case

$$\mathcal{N}(L, 0, 0) = \frac{1}{\sqrt{2\pi} 2L} \left[\frac{\langle q_1(q_1-1) \rangle}{\langle q_1 \rangle} \right]^L. \quad (5)$$

Similarly, the formula for $\mathcal{N}(0, L, 0)$ is found simply by exchanging the subscripts. On the other hand, when $s_{11} = 0$, it is still possible to have $s_{12} > 0$, when each segment in layer 1 consists of only a single edge (i.e. each switch between layers is immediately followed by another switch). Then

$$\begin{aligned} \mathcal{N}(0, s_{22}, 2p) = & (2p-1)! \frac{(s_{22}+p-1)!}{(p-1)!} \frac{e^L}{2\pi \sqrt{s_{22}s_s}} \\ & \times \langle q_1 \rangle^{-p} \langle q_2 \rangle^{-s_{22}-p} \left[\frac{\langle q_2(q_2-1) \rangle}{s_{22}} \right]^{s_{22}} \left[\frac{\langle q_1 q_2 \rangle}{s_{12}} \right]^{s_{12}}, \end{aligned} \quad (6)$$

and similarly for the case $s_{22} = 0$ but $s_{12} > 0$ and $s_{11} > 0$, by exchanging the subscripts 1 and 2. If we project the two layers onto a single network, we recover the existing result for a single colored network, which has the same form as Eq. (5).

The number of cycles having exactly $n_1 = s_{11} + \frac{1}{2}s_{12}$ edges of type 1 for a fixed L can be found by summing Eq. (4) over each s_{ab} . In the absence of inter-layer degree correlations, the resulting distributions for n_1 match the Binomial distribution found by selecting L edges at random from the network, as shown in Fig. 2. The mean number of type 1 edges is $p_1 = \langle q_1 \rangle / (\langle q_1 \rangle + \langle q_2 \rangle)$, and similarly for n_2 . In addition, the number of switches s_{12} , which must always be even, can be found by summing over s_{11} and s_{22} . The mean number of switches $\langle s_{12} \rangle$ is well predicted by $2L \langle q_1 \rangle \langle q_2 \rangle / (\langle q_1 \rangle + \langle q_2 \rangle)^2$, which is the expected number of mismatches when pairing L randomly chosen edges, and the distribution is also well matched by a binomial distribution.

The number of cycles for a given vector l in Eq. (4) depends only on the moments of the joint degree distribution. This means that networks may have quite different degree distributions, and hence different structures, but if the relevant moments are the same, so will be the average number of cycles of each kind.

In simplex networks, it has been shown that degree-degree correlations for neighboring vertices affect quite deeply the number of cycles [25]. In multiplex networks,

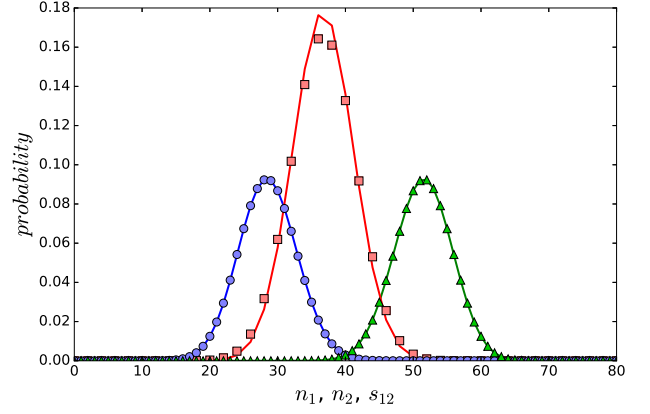


FIG. 2. Distribution of number of edges of type 1 (circles, blue online) and 2 (triangles, green online) and number of switches s_{12} (squares, red online) in cycles of length $L = 80$ in two uncorrelated layers with Poisson degree distributions having mean degrees $\langle q_1 \rangle = 25$ and $\langle q_2 \rangle = 45$. Symbols are from summation of Eq. (4), solid lines are binomial distributions for L trials with probabilities $\langle q_1 \rangle / (\langle q_1 \rangle + \langle q_2 \rangle)$, $\langle q_2 \rangle / (\langle q_1 \rangle + \langle q_2 \rangle)$, and $\langle q_1 \rangle \langle q_2 \rangle / (\langle q_1 \rangle + \langle q_2 \rangle)^2$ respectively.

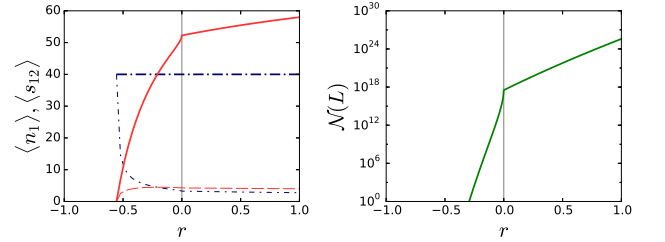


FIG. 3. (Left, color online) Mean for n_1 (blue heavy dash-dot) and s_{12} (red, solid) and corresponding standard deviations (blue light dash-dot, red light broken, respectively) as a function of Pearson correlation coefficient r for the degrees of a vertex in different layers. Assortative correlations ($r < 0$) are created using a joint degree distribution of the form $P(q_1, q_2) = \rho \frac{1}{2} [(P(q_1) + P(q_2)) \delta_{q_1, q_2} + (1 - \rho) P(q_1) P(q_2)]$. Disassortative correlations ($r < 0$) are of the form $P(q_1, q_2) = \rho [(P(q_1) \delta_{q_2, 0} + P(q_2) \delta_{q_1, 0}) + (1 - \rho) P(q_1) P(q_2)]$, where $P(q) \propto q^{-\gamma}$, with $\gamma = 3.7$, is a power-law distribution, and $L = 80$. Maximum disassortative correlation occurs when the two layers no longer overlap, at which point $n_1 = 0$. The corresponding value of r depends on γ . Figures are qualitatively similar for any value of $\gamma > 3$. (Right) Total number of cycles as a function of r .

it is natural to ask about the effect of degree correlations across layers. Indeed, we can see from the last term in Eq. (4) that interlayer degree correlations affect the number of cycles having a given number of switches between layers. For a given L , assortative interlayer degree correlations will tend to increase the number of switches, as high degree nodes in one layer, which are more frequently visited, will also have more available edges in the other layers. Conversely, disassortative correlations will tend

to decrease the number of switches. The mean number of edges of type 1 remains constant, although the distribution changes. These effects can be seen in Fig. 3, in which we plot the mean and standard deviation of n_1 and s_{12} as a function of the Pearson correlation coefficient r for the degrees of a vertex in different layers [11]. We see that in the extreme case of disassortative correlations, there are no switches and the entire cycle is of one color, as the standard deviation of n_1 reaches the maximum value of $L/2$. An even more dramatic change is seen in the total number of cycles of a given length. The right panel in Fig. 3 shows the total number of cycles of length L as a function of r . Disassortative correlations greatly restrict the possible number cycles that can be formed, while assortative correlations greatly increase it. Note that the apparently sharp inflections at $r = 0$ result simply from the use of different functional forms for assortative and disassortative correlations.

Clustering.—The clustering coefficient of a network is related to the number of triangles, that is, cycles of length three. In such short cycles, the computation of factor $G(l)$ is straightforward, thus we can calculate the number of cycles of length 3 for any number of layers. Such a cycle may be entirely within one layer: $\mathcal{N}_m^{(1)} = z_m^3/6c_m^3$; have two edges in one layer (m) and one edge in a second layer (n): $\mathcal{N}_{m,n}^{(2)} = z_m \langle q_m q_n \rangle^2 / 2c_m^2 c_n$; or have one edge each in three different layers: $\mathcal{N}_{m,n,r}^{(3)} = \langle q_m q_n \rangle \langle q_m q_r \rangle \langle q_n q_r \rangle / c_m c_n c_r$, where $z_m \equiv \langle q_m (q_m - 1) \rangle$ and $c_m = \langle q_m \rangle$.

We can then define a global clustering coefficient by

$$C = \frac{3 \sum_{l:L=3} \mathcal{N}(l)}{\mathcal{V}(N)} = \frac{3 \sum_m \mathcal{N}_m^{(1)} + 3 \sum_{m,n \neq m} \mathcal{N}_{m,n}^{(2)} + 3 \sum_{\substack{m,n \neq m \\ r \neq m,n}} \mathcal{N}_{m,n,r}^{(3)}}{N \left[\sum_m z_m + 2 \sum_{m,n \neq m} \langle q_m q_n \rangle \right]}, \quad (7)$$

where $\mathcal{V}(N)$ is the number of adjacent edge-pairs in the graph, and the summation is over all cycle vectors l having length $L = 3$.

One may also define partial clustering coefficients for triangles entirely within a given layer, two given layers, or three given layers ($C_{m,n,r}^{(3)}$) respectively:

$$C_m^{(1)} = \frac{3\mathcal{N}_m^{(1)}}{\frac{1}{2}N z_m} = \frac{z_m^2}{N c_m^3}, \quad (8)$$

$$C_{m,n}^{(2)} = \frac{3\mathcal{N}_{m,n}^{(2)}}{\frac{1}{2}N [z_m + 2\langle q_m q_n \rangle]} = \frac{3z_m \langle q_m q_n \rangle^2}{N [z_m + 2\langle q_m q_n \rangle] c_m^2 c_n}, \quad (9)$$

$$C_{m,n,r}^{(3)} = \frac{3\mathcal{N}_{m,n,r}^{(3)}}{\frac{1}{2}N [\langle q_m q_n \rangle + \langle q_m q_r \rangle + \langle q_n q_r \rangle]} \quad (10)$$

$$= \frac{6\langle q_m q_n \rangle \langle q_m q_r \rangle \langle q_n q_r \rangle}{N c_m c_n c_r [\langle q_m q_n \rangle + \langle q_m q_r \rangle + \langle q_n q_r \rangle]}, \quad (11)$$

or triangles entirely in one, two, or three layers, regardless of which particular layers they are (these less fine-grained coefficients were defined and calculated numerically in Ref. [10]):

$$C^{(1)} = \frac{\sum_m (z_m/c_m)^3}{N \sum_m z_m}, \quad (12)$$

$$C^{(2)} = \frac{3 \sum_{m,n \neq m} z_m \langle q_m q_n \rangle^2 / (c_m^2 c_n)}{N \sum_{m,n \neq m} [z_m + 2\langle q_m q_n \rangle]}, \quad (13)$$

$$C^{(3)} = \frac{6 \sum_{m,n \neq m, r \neq m,n} \langle q_m q_n \rangle \langle q_m q_r \rangle \langle q_n q_r \rangle / c_m c_n c_r}{N \sum_{m,n \neq m, r \neq m,n} \langle q_m q_n \rangle}. \quad (14)$$

These formulæ give the expected clustering coefficients for large random graphs, taking into account the full degree distribution. This gives a more accurate result than found by simply matching the mean degree to an Erdős-Rényi network [10], for which the clustering coefficients can be calculated by considering the probability for a given edge to be present or absent:

$$C_{m(ER)}^{(1)} = \frac{c_m}{N}, \quad (15)$$

$$C_{m,n(ER)}^{(2)} = \frac{3c_m c_n}{N(c_m + 2c_n)}, \quad (16)$$

$$C_{m,n,r(ER)}^{(3)} = \frac{6c_m c_n c_r}{N(c_m c_n + c_m c_r + c_n c_r)}, \quad (17)$$

$$C_{ER}^{(1)} = \frac{\sum_m c_m^3}{N \sum_m c_m^2}, \quad (18)$$

$$C_{ER}^{(2)} = \frac{3 \sum_{m,n \neq m} c_m^2 c_n}{N \sum_{m,n \neq m} (c_m^2 + 2c_m c_n)}, \quad (19)$$

$$C_{ER}^{(3)} = \frac{6 \sum_{m,n \neq m, r \neq m,n} c_m c_n c_r}{N \sum_{m,n \neq m, r \neq m,n} c_m c_n}, \quad (20)$$

$$C_{ER} = \frac{\sum_m c_m}{2N}, \quad (21)$$

which coincide with the results found by inserting uncorrelated Poisson degree distributions into Eqs. (7)-(14).

To illustrate the importance of taking correlations into account, we compared our formulæ Eqs. (7)-(14) with measurements of synthetic networks. The results are summarized in Fig. 4. This shows that our method successfully accounts for the effects of broad degree distributions, and inter-layer correlations. In comparison, the Erdős-Rényi formulæ are generally only accurate for uncorrelated Erdős-Rényi layers and fail completely in the presence of strong correlations between layers.

Conclusions.—In single layer networks, cycles are characterized by their length. In multiplex networks, there are many more possibilities. In particular, there is the possibility to switch between layers, and this must be accounted for. In this Letter we have introduced a classification for cycles in multiplex networks based on the number of edges in each layer, and the number of switches

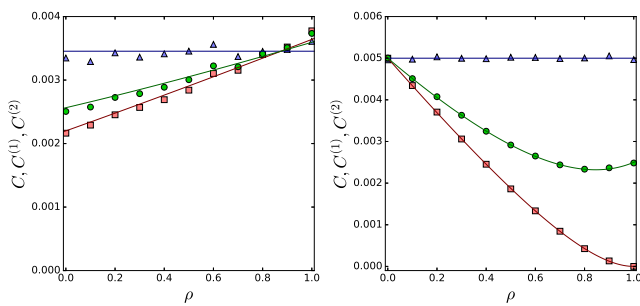


FIG. 4. (Left) Total clustering coefficient, C (circles, green online), single layer, $C^{(1)}$ (triangles, blue online) and two layer, $C^{(2)}$ (squares, red online), clustering coefficients for a two layer multiplex network with assortative inter layer degree correlations, joint degree distribution $P(q_1, q_2) = \rho \frac{1}{2} [(P(q_1) + P(q_2))\delta_{q_1, q_2} + (1 - \rho)P(q_1)P(q_2)]$, where $P(q) \propto q^{-\gamma}$, with $\gamma = 2.9$ for $q \in [10, 100]$. Each layer has $N = 10^4$ vertices. For $\rho = 0$ correlations are absent, while for $\rho = 1$ degrees are perfectly correlated. Solid lines are expected theoretical values using Eqs. (7), (12), and (13). (Right) Total C , single layer, $C^{(1)}$, and two layer, $C^{(2)}$, clustering coefficients for a two layer multiplex network with disassortative inter layer degree correlations, joint degree distribution $P(q_1, q_2) = \rho [P(q_1)\delta_{q_2, 0} + P(q_2)\delta_{q_1, 0}] + (1 - \rho)P(q_1)P(q_2)$, where $P(q)$ is a Poisson distribution with mean 50. For $\rho = 0$ correlations are absent, while for $\rho = 1$ degrees are perfectly disjoint. Solid lines are expected theoretical values using Eqs. (7), (12), and (13).

between layers. We further calculated the expected number of each type of cycle in a large random multiplex. Our results are valid for any multi-degree distribution, including distributions characterized by arbitrary degree correlations between layers. Interestingly, our formulæ show that the first and second order moments of the multi-degree distribution are sufficient to determine the statistics of cycles in large multiplex networks. The effect of correlations between a vertex's degrees in different layers affects these statistics through the degree-degree moment $\langle q_m q_n \rangle$. Assortative correlations tend to increase the number of switches in cycles of a given length, while also increasing the total number of cycles. Disassortative correlations, on the other hand, may greatly reduce both the total number of cycles and the total number of switches within these cycles.

These results further allow us to give the expected clustering coefficients in multiplex networks. The possibility that a closed triangle may have edges in multiple layers requires a more detailed discrimination of clustering coefficients. We give a complete classification of the various possible clustering coefficients and give their expected values. Inter layer correlations again have a strong effect, significantly increasing or decreasing the mixed-layer clustering coefficients. These results give a much more precise view of cycles and clustering in multiplex networks than using the mean degree alone, and estab-

lish the proper baseline for comparison with real-world networks.

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- * gjbaxter@ua.pt
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